

BOUNDS ON THE NUMBER OF REAL SOLUTIONS TO POLYNOMIAL EQUATIONS

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ABSTRACT. We use Gale duality for complete intersections and adapt the proof of the fewnomial bound for positive solutions to obtain the bound

$$\frac{e^4 + 3}{4} 2^{\binom{k}{2}} n^k$$

for the number of non-zero real solutions to a system of n polynomials in n variables having $n+k+1$ monomials whose exponent vectors generate a subgroup of \mathbb{Z}^n of odd index. This bound only exceeds the bound for positive solutions by the constant factor $(e^4 + 3)/(e^2 + 3)$ and it is asymptotically sharp for k fixed and n large.

INTRODUCTION

In [3], the sharp bound of $2n+1$ was obtained for the number of non-zero real solutions to a system of n polynomial equations in n variables having $n+2$ monomials whose exponents affinely span the lattice \mathbb{Z}^n . In [4], the sharp bound of $n+1$ was given for the positive solutions to such a system of equations. This last bound was generalized in [7], which showed that the number of positive solutions to a system of n polynomial equations in n variables having $n+k+1$ monomials was less than

$$\frac{e^2 + 3}{4} 2^{\binom{k}{2}} n^k,$$

which is asymptotically sharp for k fixed and n large [5]. This dramatically improved Khovanskii's fewnomial bound [8] of $2^{\binom{n+k}{2}} (n+1)^{n+k}$.

We give a bound for all non-zero real solutions. Under the assumption that the exponent vectors \mathcal{W} span a subgroup of \mathbb{Z}^n of odd index, we show that the number of non-degenerate non-zero real solutions to a system of polynomials with support \mathcal{W} is less than

$$(1) \quad \frac{e^4 + 3}{4} 2^{\binom{k}{2}} n^k.$$

The novelty is that this bound exceeds the bound for solutions in the positive orthant by a fixed constant factor $(e^4 + 3)/(e^2 + 3)$, rather than by a factor of 2^n , which is the number of orthants. By the construction in [5], it is asymptotically sharp for k fixed and n large.

We follow the outline of [7]—we use Gale duality for real complete intersections [6] and then bound the number of solutions to the dual system of master functions. The

2000 *Mathematics Subject Classification.* 14M25, 14P25, 52C35.

Key words and phrases. sparse polynomial system, hyperplane arrangement, fewnomial.

Bates and Sottile supported by the Institute for Mathematics and its Applications.

Sottile supported by the NSF CAREER grant DMS-0538734.

key idea is that including solutions in all chambers in a complement of an arrangement of hyperplanes in \mathbb{RP}^k , rather than in just one chamber as in [7], does not increase our estimate on the number of solutions very much. This was discovered while implementing a numerical continuation algorithm for computing the positive solutions to a system of polynomials [1]. That algorithm was improved by this discovery to one which finds all real solutions. It does so without computing complex solutions and is based on [7] and the results of this paper. Its complexity depends on (1), and not on the number of complex solutions.

We state our main theorem in Section 1 and then use Gale duality to reduce it to a statement about systems of master functions, which we prove in Section 2.

1. GALE DUALITY FOR SYSTEMS OF SPARSE POLYNOMIALS

Let $\mathcal{W} = \{w_0 = 0, w_1, \dots, w_{n+k}\} \subset \mathbb{Z}^n$ be a collection of $n+k+1$ integer vectors ($|\mathcal{W}| = n+k+1$), which correspond to monomials in variables x_1, \dots, x_n . A (Laurent) polynomial f with *support* \mathcal{W} is a real linear combination of monomials with exponents from \mathcal{W} ,

$$(2) \quad f(x_1, \dots, x_n) = \sum_{i=0}^{n+k} c_i x^{w_i} \quad \text{with } c_i \in \mathbb{R}.$$

A *system with support* \mathcal{W} is a system of polynomial equations

$$(3) \quad f_1(x_1, \dots, x_n) = f_2(x_1, \dots, x_n) = \dots = f_n(x_1, \dots, x_n) = 0,$$

where each polynomial f_i has support \mathcal{W} . Since multiplying every polynomial in (3) by a monomial x^α does not change the set of non-zero solutions but translates \mathcal{W} by the vector α , we see that it was no loss of generality to assume that $0 \in \mathcal{W}$.

The system (3) has infinitely many solutions if \mathcal{W} does not span \mathbb{R}^n . We say that \mathcal{W} *spans $\mathbb{Z}^n \bmod 2$* if the \mathbb{Z} -linear span of \mathcal{W} is a subgroup of \mathbb{Z}^n of odd index.

Theorem 1. *Suppose that \mathcal{W} spans $\mathbb{Z}^n \bmod 2$ and $|\mathcal{W}| = n+k+1$. Then there are fewer than (1) non-degenerate non-zero real solutions to a sparse system (3) with support \mathcal{W} .*

The importance of this bound for the number of real solutions is that it has a completely different character than Kouchnirenko's bound for the number of complex solutions.

Proposition 2 (Kouchnirenko [2]). *The number of non-degenerate solutions in $(\mathbb{C}^\times)^n$ to a system (3) with support \mathcal{W} is no more than $n! \text{vol}(\text{conv}(\mathcal{W}))$.*

Here, $\text{vol}(\text{conv}(\mathcal{W}))$ is the Euclidean volume of the convex hull of \mathcal{W} .

Perturbing coefficients of the polynomials in (3) so that they define a complete intersection in $(\mathbb{C}^\times)^n$ can only increase the number of non-degenerate solutions. Thus it suffices to prove Theorem 1 under this assumption. Such a complete intersection is equivalent to a complete intersection of master functions in a hyperplane complement [6].

Let \mathbb{R}^{n+k} have coordinates z_1, \dots, z_{n+k} . A polynomial (2) with support \mathcal{W} is the pull-back $\Phi_{\mathcal{W}}^*(\Lambda)$ of the degree 1 polynomial $\Lambda := c_0 + c_1 z_1 + \dots + c_{n+k} z_{n+k}$ along the map

$$\Phi_{\mathcal{W}} : (\mathbb{R}^\times)^n \ni x \longmapsto (x^{w_i} \mid i = 1, \dots, n+k) \in \mathbb{R}^{n+k}.$$

If we let $\Lambda_1, \dots, \Lambda_n$ be the degree 1 polynomials which pull back to the polynomials in the system (3), then they cut out an affine subspace L of \mathbb{R}^{n+k} of dimension k .

Let $\{p_i \mid i = 1, \dots, n+k\}$ be degree 1 polynomials on \mathbb{R}^k which induce an isomorphism between \mathbb{R}^k and L ,

$$\Psi_p : \mathbb{R}^k \ni y \longmapsto (p_1(y), \dots, p_{n+k}(y)) \in L \subset \mathbb{R}^{n+k}.$$

Let $\mathcal{A} \subset \mathbb{R}^k$ be the arrangement of hyperplanes defined by the vanishing of the $p_i(y)$. This is the pullback along Ψ_p of the coordinate hyperplanes of \mathbb{R}^{n+k} .

The image $\Phi_{\mathcal{W}}((\mathbb{R}^\times)^n)$ inside of the torus $(\mathbb{R}^\times)^{n+k}$ has equations

$$z^{\beta_1} = z^{\beta_2} = \dots = z^{\beta_k} = 1,$$

where the *weights* $\{\beta_1, \dots, \beta_k\}$ form a basis for the \mathbb{Z} -submodule of \mathbb{Z}^{n+k} of linear relations among the vectors \mathcal{W} . To these data, we associate a system of *master functions* on the complement $M_{\mathcal{A}}$ of the arrangement \mathcal{A} of \mathbb{R}^k ,

$$(4) \quad p(y)^{\beta_1} = p(y)^{\beta_2} = \dots = p(y)^{\beta_k} = 1.$$

Here, if $\beta = (b_1, \dots, b_{n+k})$ then $p^\beta := p_1(y)^{b_1} \dots p_{n+k}(y)^{b_{n+k}}$.

A basic result of [6] is that if \mathcal{W} spans \mathbb{Z}^n modulo 2 and either of the systems (3) or (4) defines a complete intersection, then the other defines a complete intersection and the maps $\Phi_{\mathcal{W}}$ and Ψ_p induce isomorphisms between the two solution sets, as analytic subschemes of $(\mathbb{R}^\times)^n$ and $M_{\mathcal{A}}$. Since we assumed that the system (3) is general, these hypotheses hold and the arrangement is *essential* in that the polynomials p_i span the space of all degree 1 polynomials on \mathbb{R}^k .

Theorem 3. *A system (4) of master functions in the complement of an essential arrangement of $n+k$ hyperplanes in \mathbb{R}^k has at most (1) non-degenerate real solutions.*

We actually prove a bound for a more general system than (4), namely for

$$p(z)^{2\beta_1} = p(z)^{2\beta_2} = \dots = p(z)^{2\beta_k} = 1.$$

We write this more general system as

$$(5) \quad |p(z)|^{\beta_1} = |p(z)|^{\beta_2} = \dots = |p(z)|^{\beta_k} = 1.$$

In a system of this form we may have real number weights $\beta_i \in \mathbb{R}^{n+k}$. We give the strongest form of our theorem.

Theorem 4. *A system of the form (5) with real weights β_i in the complement of an essential arrangement of $n+k$ hyperplanes in \mathbb{R}^k has at most (1) non-degenerate real solutions.*

2. PROOF OF THEOREM 4

We follow [7] with minor, but important, modifications. Perturbing the polynomials $p_i(y)$ and the weights β_j will not decrease the number of non-degenerate real solutions in $M_{\mathcal{A}}$. This enables us to make the following assumptions.

The arrangement $\mathcal{A}^+ \subset \mathbb{RP}^k$, where we add the hyperplane at infinity, is general in that every j hyperplanes of \mathcal{A}^+ meet in a $(k-j)$ dimensional linear subspace, called a *codimension j face of \mathcal{A}* . If B is the matrix whose columns are the weights β_1, \dots, β_k , then the entries of B are rational numbers and no minor of B vanishes. This last technical

condition as well as the freedom to further perturb the β_j and the p_i are necessary for the results in [7, Section 3] upon which we rely.

For functions f_1, \dots, f_j on $M_{\mathcal{A}}$, let $V(f_1, \dots, f_j)$ be the subvariety they define. Suppose that $\beta_j = (b_{1,j}, \dots, b_{n+k,j})$. For each $j = 1, \dots, k$, define

$$\psi_j(y) := \sum_{i=1}^{n+k} b_{i,j} \log |p_i(y)|.$$

Then (5) is equivalent to $\psi_1(y) = \dots = \psi_k(y) = 0$. Inductively define $\Gamma_k, \Gamma_{k-1}, \dots, \Gamma_1$ by

$$\Gamma_j := \text{Jac}(\psi_1, \dots, \psi_j, \Gamma_{j+1}, \dots, \Gamma_k),$$

the Jacobian determinant of $\psi_1, \dots, \psi_j, \Gamma_{j+1}, \dots, \Gamma_k$. Set

$$C_j := V(\psi_1, \dots, \psi_{j-1}, \Gamma_{j+1}, \dots, \Gamma_k),$$

which is a curve in $M_{\mathcal{A}}$.

Let $\mathfrak{b}(C)$ be the number of unbounded components of a curve $C \subset M_{\mathcal{A}}$. We have the estimate from [7], which is a consequence of the Khovanskii-Rolle Theorem,

$$(6) \quad |V(\psi_1, \dots, \psi_k)| \leq \mathfrak{b}(C_k) + \dots + \mathfrak{b}(C_1) + |V(\Gamma_1, \dots, \Gamma_k)|.$$

Here, $|S|$ is the cardinalty of the set S . We estimate these quantities.

Lemma 5.

- (1) $|V(\Gamma_1, \dots, \Gamma_k)| \leq 2^{\binom{k}{2}} n^k$.
- (2) C_j is a smooth curve and

$$\mathfrak{b}(C_j) \leq \frac{1}{2} 2^{\binom{k-j}{2}} n^{k-j} \binom{n+k+1}{j} \cdot 2^j \leq \frac{1}{2} 2^{\binom{k}{2}} n^k \cdot \frac{2^{2j-1}}{j!}.$$

Proof of Theorem 4. By (6) and Lemma 5, we have

$$|V(\psi_1, \dots, \psi_k)| \leq 2^{\binom{k}{2}} n^k \left(1 + \frac{1}{4} \sum_{j=1}^k \frac{4^j}{j!} \right) < 2^{\binom{k}{2}} n^k \cdot \frac{e^4 + 3}{4}. \quad \square$$

Proof of Lemma 5. The bound (1) is from Lemma 3.4 of [7]. Statements analogous to (2) for \tilde{C}_j , the restriction of C_j to a single chamber (connected component) of $M_{\mathcal{A}}$, were established in Lemma 3.4 and the proof of Lemma 3.5 in [7]:

$$(7) \quad \mathfrak{b}(\tilde{C}_j) \leq \frac{1}{2} 2^{\binom{k-j}{2}} n^{k-j} \binom{n+k+1}{j} \leq \frac{1}{2} 2^{\binom{k}{2}} n^k \cdot \frac{2^{j-1}}{j!}.$$

The bound we claim for $\mathfrak{b}(C_j)$ has an extra factor of 2^j . *A priori* we would expect to multiply this bound (7) by the number of chambers of $M_{\mathcal{A}}$ to obtain a bound for $\mathfrak{b}(C_j)$, but the correct factor is only 2^j .

We work in \mathbb{RP}^k and use the extended hyperplane arrangement \mathcal{A}^+ , as we will need points in the closure of C_j in \mathbb{RP}^k . The first inequality in (7) for $\mathfrak{b}(\tilde{C}_j)$ arises as each

unbounded component of \tilde{C}_j meets \mathcal{A}^+ in two distinct points (this accounts for the factor $\frac{1}{2}$) which are points of codimension j faces where the polynomials

$$F_i(y) := \Gamma_{k-i}(y) \cdot \left(\prod_{i=1}^{n+k} p_i(y) \right)^{2^i}$$

for $i = 0, \dots, k-j-1$ vanish. (By Lemma 3.4(1) of [7], F_i is a polynomial of degree $2^i n$.) The genericity of the weights and the linear polynomials $p_i(y)$ imply that these points will lie on faces of codimension j but not of higher codimension. The factor $2^{\binom{k-j}{2}} n^{k-j}$ is the Bézout number of the system $F_0 = \dots = F_{k-j-1}$ on a given codimension j plane, and there are exactly $\binom{n+k+1}{j}$ codimension j faces of \mathcal{A}^+ .

At each of these points, C_j will have one branch in each chamber of $M_{\mathcal{A}}$ incident on that point. Since the hyperplane arrangement \mathcal{A}^+ is general there will be exactly 2^j such chambers. \square

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